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Functional Integration for Bose Fields

M. Revzen[1] and L. S. Schulman[1]

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The Lagrangian occurring in functional integration of bose fields is shown to be made up of two parts, one pertaining to the Bose commutation and the other to the dynamics.

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In field-theoretic(1) and statistical physics(2,3) one may replace the Hamiltonian formalism by a Lagrangian formalism. The advantage of the latter is the use of c-number functions. In this note we interpret the functional integral to be composed of two physically distinct parts. One merely generates the commutation of equal time operators(4) and the other provides the dynamics. The appreciation of this fact may clarify the use of the timelike parameter appearing in the functional integration formalism and show that it need not be identified with a dynamic parameter (i.e., time or inverse temperature).

Consider the matrix element between two coherent states(5) ($\hbar = 1$)

$$\begin{aligned}&\langle \alpha \mid a[\exp(iHt)]\, a^{+} \mid \alpha' \rangle \\ &\quad = \int d\alpha(s) \exp\left\{\int_0^1 [\tfrac{1}{2}(\alpha^*\dot{\alpha} - \dot{\alpha}^*\alpha) - itH(\alpha^*, \alpha)]\, ds\right\} \alpha(s_1)\, \alpha^*(s_2) \qquad (1)\end{aligned}$$

with $\alpha(0) = \alpha'$, $\alpha(1) = \alpha$, $s_1 > s_2$, and $\dot{\alpha} = \partial\alpha/\partial s$. We now consider $t = 0$, thereby eliminating the dynamic part, and see directly that the remainder

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[1] Department of Physics, Technion-Israel Institute of Technology, Haifa, Israel.

affects the commutation relation. Thus the expression in full is [$s_1 = (\nu + 1)$ and $s_2 = \lambda$]

$$\langle \alpha \mid \alpha_1 \rangle \langle \alpha_1 \mid \cdots \mid \alpha_\nu \rangle \langle \alpha_\nu \mid \alpha_{\nu+1} \mid \alpha_{\nu+1} \rangle \cdots \langle \alpha_\lambda \mid \alpha_\lambda{}^* \mid \alpha_{\lambda+1} \rangle \cdots \langle \alpha_N \mid \alpha \rangle \quad (2)$$

Here the scalar product

$$\langle \alpha_n \mid \alpha_{n+1} \rangle = \exp[-(\mid \alpha_n \mid^2 + \mid \alpha_{n+1} \mid^2)/2 + \alpha_n{}^* \alpha_{n+1}],$$

and the unit operator $\mid \alpha \rangle \langle \alpha \mid$ designates integration $\int d^2\alpha \, \pi^{-1} \mid \alpha \rangle \langle \alpha \mid$. It is easy to see that this expression is equivalent to $\langle \alpha \mid aa^+ \mid \alpha' \rangle$. To see the generation of the commutation relation, we note that a typical integral in expression (2) is

$$\int d^2\alpha_\lambda \, \pi^{-1} \exp(- \mid \alpha_\lambda \mid^2 + \alpha_\lambda{}^* \alpha_{\lambda+1} + \alpha_{\lambda-1}^* \alpha_\lambda) \, \alpha_\lambda{}^* = \alpha_{\lambda-1}^* \exp(\alpha_{\lambda+1}^* \alpha_{\lambda-1})$$

Thus $\alpha_\lambda{}^*$ moves to $\alpha_{\lambda-1}^*$. Similarly α_ν moves to the right. When the two indices become equal the integral yields $(1 + \alpha_{n-1}^* \alpha_{n+1}) \exp(\alpha_{n+1}^* \alpha_{n-1})$. Continuing we get (aside from a factor $\langle \alpha \mid \alpha' \rangle$) $1 + \alpha^* \alpha'$. In exactly the same way $\langle \alpha \mid a^+ a \mid \alpha' \rangle$ gives $\alpha^* \alpha'$. Hence the dynamically independent term generates the commutation relation. We thus interpret the terms in the Lagrangian of the functional integral of Eq. (1) as: (i) establishing proper commutation,[4] and (ii) providing dynamics—we emphasize that the dynamics could involve interactions.

To see the connection between the above and Planck's constant, we return to Eq. (1). Here $\hbar$ had been absorbed into the α's. Redefining $\alpha \to \gamma = \sqrt{\hbar} \alpha$ and noting that the energy per mode and the energy of interaction *do not* depend on the normalization of the amplitudes of the modes, we get as the *sole* modification in Eq. (1) $\alpha^* \dot{\alpha} - \dot{\alpha}^* \alpha \to \hbar^{-1}(\gamma^* \dot{\gamma} - \dot{\gamma}^* \gamma)$.[2] In the limit $\hbar \to 0$ this term is the only one affected—it is the term which we previously identified as giving the commutation rules. This term is pure imaginary, so that the classical limit is obtained in the asymptotic sense from the stationary phase approximation, i.e., $\dot{\gamma} = 0$. Thus the *expectation* value of the energy in one mode goes over to the classical limit when the s dependence of γ is dropped.

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[2] The appearance of $\hbar$ in the Jacobian is neglected here; it cancels in statistical mechanical expectation values.

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